Proof of the finite-time thermodynamic uncertainty relation for steady-state currents

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(Rceived 12 July 2017; published 25 August 2017)

The thermodynamic uncertainty relation offers a universal energetic constraint on the relative magnitude of current fluctuations in nonequilibrium steady states. However, it has only been derived for long observation times. Here, we prove a recently conjectured finite-time thermodynamic uncertainty relation for steady-state current fluctuations. Our proof is based on a quadratic bound to the large deviation rate function for currents in the limit of a large ensemble of many copies.

DOI: 10.1103/PhysRevE.96.020103

Introduction. The thermodynamic uncertainty relation offers a fundamental bound on the current fluctuations in nonequilibrium steady states [1–4]. Roughly speaking, it states that small fluctuations come at the cost of more dissipation. This relation, and its cousins [5,6], allows one to constrain thermodynamic forces in enzymatic catalysis [7,8], bound the power fluctuations in mesoscopic machines [9,10], and limit the energetic cost of sensing [11], and it has been adapted to Brownian motion [4,12], nonequilibrium self-assembly [13], active matter [14], equilibrium order parameter fluctuations [15], activity fluctuations [16], and first-passage-time fluctuations [16,17]. Its derivation relies on bounding the likelihood of rare current fluctuations in the long-time limit, using the tools of large deviation theory [18]. As such, the predictions are proved to be valid only for long observation times [3].

Recently, though, Pietzonka et al. conjectured a steady-state-currents uncertainty relation valid for finite observation times, based on extensive numerical and experimental evidence [19]. This finite-time uncertainty relation stipulates that in the steady state any current \( J_T \) integrated up to time \( T \) will have a variance \( \text{Var}(J_T) \) and mean \( \langle J_T \rangle \) constrained by the total steady-state entropy production \( \Sigma_T \) accumulated by \( T \) as

\[
\frac{\text{Var}(J_T)}{\langle J_T \rangle^2} \geq \frac{2}{\Sigma_T},
\]

in units where Boltzmann’s constant is set to \( k_B = 1 \). For the special case of the fluctuating entropy production \( \Sigma_T \), the uncertainty relation simplifies to

\[
\text{Var}(\Sigma_T) \geq 2\langle \Sigma_T \rangle,
\]

which has been derived directly from the structure of entropy-production fluctuations for nonequilibrium systems modeled as diffusion processes [20]. In this Rapid Communication, we provide a proof of Eq. (1) using the tools of large deviation theory in a manner akin to the proof of the original long-time uncertainty relation [3]. This proof puts the finite-time uncertainty relation on firm footing, justifying its use in analyzing even short-time experimental data.

Setup. We have in mind a nonequilibrium system with states \( x = 1, \ldots, M \). Transitions between states, say from \( z \) to \( y \), are modeled as a continuous-time Markov jump process with rates \( r_{yz} \). We assume that the matrix of transition rates is irreducible, so that there is a unique steady-state distribution \( \pi \) with steady-state current \( j^\pi_{yz} = r_{yz}\pi_z - r_{zy}\pi_y \). In addition, we assume that the transition rates are thermodynamically consistent, so that every ratio of transition rates can be related to a thermodynamic force \( F_{yz} = \ln(r_{yz}\pi_z/r_{zy}\pi_y) \), which measures the total entropy production—environmental entropy flow and change in system Shannon entropy—along that transition [21].

Now, as we track a stochastic realization of our system evolving over a finite time interval \( t \in [0,T] \), \( x(t) \), there will be a fluctuating instantaneous current counting every time \( t_k \) the system jumps:

\[
j_{yz}(t) = \sum_k \delta(t - t_k)(\delta(x(t_k^+),y)\delta(x(t_k^+),z) - \delta(x(t_k^-),z)\delta(x(t_k^-),y)),
\]

with \( x(t_k^\pm) \) being the state of the system just before and after a jump. Our interest, though, is in integrated generalized currents, which are obtained by weighing each mesoscopic jump by a factor \( d_{yz}(t) = -d_{zy}(t) \) and summing them up:

\[
J_T = \int_0^T ds \sum_{y<z} d_{yz}(s)j_{yz}(s).
\]

For example, the entropy production is a generalized current with \( d_{yz} = F_{yz} \),

\[
\Sigma_T = \int_0^T ds \sum_{y<z} F_{yz}j_{yz}(s),
\]

whose steady-state average

\[
\langle \Sigma_T \rangle = T \sum_{y<z} F_{yz}j^\pi_{yz} \equiv T \Sigma^\pi
\]

characterizes the irreversibility of the nonequilibrium steady state. Our goal now is to constrain the fluctuations in \( J_T \) by bounding its large deviation rate function using \( \langle \Sigma_T \rangle = T \Sigma^\pi \), which will lead to (1).

Large deviations for large ensembles. Imagine now not just one instance of our system hopping among its states, but an ensemble of \( N \gg 1 \) independent copies—labeled \( x^\alpha(t) \), \( \alpha = 1, \ldots, N \)—with initial conditions sampled from the steady-state distribution \( \pi \). Then in any given moment we could obtain an empirical estimate of the density to be in mesostate \( y \) at time \( t \) by measuring the instantaneous fraction of copies in state \( y \):

\[
\rho_y(t) = \frac{1}{N} \sum_{\alpha=1}^N \delta(x^\alpha(t),y).
\]
We could additionally estimate the current by counting the total net number of jumps along any link as

$$\phi_{yz}(t) = \frac{1}{N} \sum_{a=1}^{N} j_{za}^a(t),$$

(8)

with $j_{za}^a(t)$ the instantaneous current of copy $a$ [cf. (3)]. Indeed, the law of the large numbers guarantees that both empirical measures converge to their expected values as $N \to \infty$. However, we can also quantify their fluctuations through a large deviation principle. As demonstrated in Ref. [22], the large deviation rate function is

$$I[\rho(t), \phi(t)] = \int_0^T ds \mathcal{I}(\rho(s), \phi(s)) - S(\rho(0)||\pi).$$

(10)

The second term is the relative entropy between the initial fluctuating density $\rho(0)$ and the steady state $\pi$, $S(\rho(0)||\pi) = \sum_t \rho_t(0) \ln (\rho_t(0)/\pi_t)$. The first term can be put in the form [23,24]

$$\mathcal{I}(\rho(t), \phi(t)) = \sum_{y<z} \Psi(j_{yz}(t), j_{zy}(t), a_{yz}^t(t))$$

(11)

with

$$\Psi(j, \bar{j}, a) = j \left( \frac{\arcsin j}{a} - \frac{\arcsin \bar{j}}{a} \right) - \left( \sqrt{a^2 + j^2} - \sqrt{a^2 + \bar{j}^2} \right),$$

(12)

$j_{yz}^t(t) = r_{zy} \rho_z(t) - r_{zy} \rho_y(t)$ the expected current for density $\rho$, and $a_{yz}^t(t) = 2 \sqrt{\rho_z(t) \rho_y(t) r_{zy}}$. The expression for $I$ only applies for fluctuations that conserve probability, $\rho_t(t) = \sum_{y,z} \phi_{yz}(t)$ with a normalized density $\sum_y \rho_y(t) = 1$; otherwise, $I$ is infinity.

Within this framework, the fluctuations in the generalized current are simply due to the sum over the fluctuations of each member:

$$\Phi_d = \sum_{a=1}^{N} \left( \int_0^T ds \sum_{y<z} d_{czy}(s) j_{zy}^a(s) \right)$$

(13)

$$= N \int_0^T ds \sum_{y<z} d_{czy}(s) \phi_{yz}(t) \equiv N \Phi_d.$$  

(14)

Importantly, the large-$N$ scaling of the cumulants of $\Phi_d$ are identical to the cumulants of our generalized current $J_T$ [cf. (4)] of interest,

$$\lim_{N \to \infty} \frac{1}{N} \text{Var}(\Phi_d) = \text{Var}(J_T)$$

$$\lim_{N \to \infty} \frac{1}{N} \langle \Phi_d \rangle = \langle J_T \rangle,$$

(15)

since our ensemble of copies are independent and identically distributed. Furthermore, they are encoded in the large deviation rate function $I(\phi_d)$ for the generalized current.

Thus, by bounding $I(\phi_d)$, as we now do, we constrain the generalized-current fluctuations.

**Bounding the large deviation rate function.** Remarkably, $\mathcal{I}$ in (11) has the exact same functional form as the level-2.5 large deviation rate function for long-time-averaged empirical density and currents [23–25]. As a consequence, we can almost directly import the proof used to derive the long-time thermodynamic uncertainty relation to this situation. As such we proceed in two steps [3]: First, we bound $\mathcal{I}$, and then exploit the large-deviation contraction principle to obtain an inequality for the rate function $I(\phi_d)$.

As shown in Refs. [3,4], $\mathcal{I}$ satisfies a quadratic inequality, which in this situation reads

$$\int_0^T ds \sum_{y<z} [\phi_{yz}(s) - j_{zy}^0(s)]^2 \sigma_{yz}^2(s)$$

$$- S(\rho(0)||\pi),$$

(16)

where $\sigma_{yz}^2(s) = j_{zy}^0(s) \ln [r_{zy} \rho_y(s)/r_{zy} \rho_z(s)]$ is the expected entropy production along jump $z \to y$ if the density were $\rho$.

The next step is to contract down to the large deviation rate function for generalized current. Namely, we can obtain the large deviation function for generalized current through the minimization [18]:

$$I(\phi_d) = \inf_{\rho(0),\phi(0)} I[\rho(t), \phi(t)],$$

(17)

where the minimization is constrained by $\phi_d = \int_0^T ds \sum_{y<z} d_{czy}(s) \phi_{yz}(s)$, the conservation of probability $\rho_t(t) = \sum_{y,z} \phi_{yz}(t)$, and normalization $\sum_y \rho_y(t) = 1$. However, an upper bound to such a minimization can be obtained by choosing any pair of $\rho$ and $\phi$ consistent with the constraints. We choose the time-independent pair

$$\rho_z(t) = \pi_z, \quad \phi_{yz}(t) = \frac{\phi_d}{(JT)} j_{zy}^t.$$

(18)

Substituting into (17), while exploiting (16), we obtain the quadratic bound

$$I(\phi_d) \leq \frac{(\phi_d - (JT))^2}{4(JT)^2} \int_0^T ds \sum_{y<z} \sigma_{yz}^2$$

(19)

$$= \frac{(\phi_d - (JT))^2}{(4JT)^2} (\Sigma_T),$$

(20)

in terms of the time-integrated steady-state entropy production $\Sigma_T = T \Sigma = T \sum_{y<z} \sigma_{yz}^2$.

The finite-time uncertainty relation (1) now follows readily, by observing that the quadratic bound is zero at the typical value, $I((JT)) = 0$, and that the second derivative of $I(\phi_d)$ at its minimum encodes the large-$N$ scaling of the variance:

$$\lim_{N \to \infty} \frac{1}{N} \text{Var}(\Phi_d) = \frac{1}{I''((JT))} \geq 2(\text{JT})^2/\Sigma_T,$$

(21)

by (20). Combining this inequality with the independent-identically-distributed nature of the copies (15) leads to the thermodynamic uncertainty relation in (1).

**Discussion.** Remarkably, the finite-time uncertainty relation can be derived in almost the exact same manner as the long-time uncertainty relation using a large deviation theory for an ensemble of many copies. Consequently, this
finite-time uncertainty relation is expected to also hold for diffusion processes, since the large deviation function for diffusions has a quadratic structure identical to (16) [4,26]. Similarly, we expect that tighter-than-quadratic bounds [5,7] will also hold for finite times. Extending these constructions to an uncertainty relation for finite-time first-passage-time fluctuations would be an interesting and useful extension (cf. [17]). However, an extension to a discrete-time process appears untenable [27].

Acknowledgments. We gratefully acknowledge the Gordon and Betty Moore Foundation for supporting T.R.G. and J.M.H. through Grant No. GBMF4513.